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MORE ON MONEY AS A MEDIUM OF EXCHANGE

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ABSTRACT

We extend the analysis of Kiyotaki and Wright, who study an economy in which the different commodities that serve as media of exchange are determined endogenously. Kiyotaki and Wright consider only symmetric, steady-state, pure-strategy equilibria, and find that for some parameter values no such equilibria exist. We consider mixed-strategy equilibria and dynamic equilibria. We prove that a steady-state equilibrium exists for all parameter values and that the number of steady-state equilibria is generically finite. We also show, however, that there may be a continuum of dynamic equilibria. Further, some dynamic equilibria display cycles.

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1. Introduction

Kiyotaki and Wright (1989) present a model economy in which the objects that serve as media of exchange, or monies, are determined endogenously. Which objects serve as money depends on their intrinsic properties and, also, on the extrinsic beliefs of the agents in the model. In one version, called Model A, different objects can play the role of commodity money, depending on parameters. In fact, parameter space can be partitioned into three regions: in one region there is a unique symmetric, steady-state, pure-strategy equilibrium with a single commodity money; in the second region there is a unique such equilibrium with two commodity monies; and in the third region no symmetric, steady-state, pure-strategy equilibria exist. In another version of the model, called Model B, there is a symmetric, steady-state, pure-strategy equilibrium for all parameter values, and for some parameters there simultaneously exists a second such equilibrium with different commodity monies.

We extend the analysis in Kiyotaki and Wright (1989) to include mixed-strategy equilibria (which can also be interpreted as pure-strategy but nonsymmetric equilibria), and to include dynamic (that is, not necessarily steady-state) equilibria. Allowing for mixed-strategies allows us to do several things. First, we construct a symmetric, steady-state equilibrium in the region of parameter space for which no symmetric, steady-state, pure-strategy equilibrium exists. This is not too surprising, since Aiyagari and Wallace (1989) prove that there always exists a symmetric, mixed-strategy, steady-state equilibrium in a fairly general version of the model. Yet we think that it is useful to have a constructive proof that delivers the equilibrium strategies and commodity monies explicitly; as Aiyagari and Wallace argue, establishing existence is relatively easy, and the hard part is determining the characteristics of an equilibrium, such as if and how agents trade. Further, the equilibrium we construct is a natural one in the sense that it continuously "ties together" the pure-strategy equilibria found earlier.

Second, we show that once we allow mixed-strategies we lose uniqueness in Model A, by constructing a new mixed-strategy, steady-state equilibrium in a nonempty open set of parameter space for which we already know that a pure-strategy, steady-state equilibrium exists. The economics of these two equilibria are rather different and, in particular, they are characterized by different commodity monies, simply because agents have different extrinsic beliefs as to what the commodity monies will be. This leads us to address the issue of exactly how many steady-state equilibria there might be in the model. We prove that generically there is a finite number of steady-state equilibria in either Model A or Model B. This means that extrinsic beliefs do not allow just anything to happen, and is also important because it allows us to do comparative statics and other natural experiments (see, for example, Kehoe 1985).

When we extend the analysis beyond steady-states, however, the set of equilibria becomes very rich indeed. We first demonstrate that, at least for certain regions of parameter space, there can be a robust continuum of equilibrium paths. That is, for certain nonempty open sets in parameter space, there is a stationary steady-state equilibrium, and for any initial values of the (predetermined) inventories in some nonempty open set, there exists a continuum of strategies such that the inventories and strategies converge to the steady-state along a path starting from these initial conditions. Hence, given the predetermined inventory distribution, there is a continuum of initial conditions for trading strategies all of which are consistent with a dynamic equilibrium. We also show that the model can display equilibria that are stable limit cycles. In these equilibria, the trading strategies and, therefore, the commodity monies vary cyclically, even though the fundamentals of the model are time invariant.

It is worth remarking that the dynamic equilibria of this model are reminiscent of the dynamic equilibria in overlapping generations economies. Further, we exploit the same types of tools that are used in the study of those economies. For example, using similar techniques, Kehoe and Levine (1984) demonstrate the generic finiteness of the number of steady-state equilibria in overlapping generations economies; Kehoe and Levine (1985) analyze the possibility of a continuum of dynamic equilibria converging to a steady-state; and Benhabib and Day (1982), Grandmont (1985), and Azariadis and Geusnerie (1986) analyze the possibility of equilibrium cycles and even more complex dynamics. It is also worth pointing out that our results are not special cases of general theorems in game theory, for reasons that we discuss further below. In particular, the model we analyze is an anonymous sequential game (see Jovanovic and Rosenthal 1988); hence, for example, very little about the set of equilibria can be learned from the literature on repeated games, as "reputation" is irrelevant here.

The remainder of the paper is organized as follows. In Section 2 we review the basic model in Kiyotaki and Wright (1989), and state the results on the existence of a symmetric, steady-state, pure-strategy, commodity money equilibrium. (That paper also considered economies with fiat money, but here we restrict attention to the commodity money case.) As mentioned earlier, there is a region of parameter space for which no such equilibrium exists. In Section 3 we extend the model to mixed-strategies and dynamics. In Section 4 we fill in the gap in the results of Section 2 by constructing a steady-state equilibrium in mixed-strategies. We also construct a mixed-strategy steady-state equilibrium for parameters for which we already know that a pure-strategy equilibrium exists. In Section 5 we prove that the number of steady-state equilibria is generically finite. In Section 6 we demonstrate that a robust continuum of dynamic equilibria can arise, and in Section 7 we construct a cyclic equilibrium. In Section 8 we present some brief concluding remarks.

2. The Basic Model

Time is discrete and continues forever. There are three indivisible goods, labeled i = 1, 2, 3. There is a continuum of agents of unit mass, with equal proportions of three types: type i consumes only good i and produces only good i+1 modulo 3 (type 1 produces good 2, type 2 produces good 3, and type 3 produces good 1). For each type i, u is the utility of consuming good i, and c_{ij} is the disutility of storing good j for one period. The cost of production is normalized to zero, and $\beta\in(0,1)$ is the discount factor. Assume that only one good at a time can be stored, and assume for now that storage costs are ordered $c_{i3} > c_{i2} > c_{i1}$ for all i. This is the consumption-production-storage specification called Model A in Kiyotaki and Wright (1989). Model B reorders production so that i produces i-1 modulo 3 or, equivalently, reorders storage costs. It will be more convenient here to have i always produce i+1, and to differentiate Models A and B by reordering storage costs.

Agents meet randomly in pairs at each date and trade bilaterally if and only if it is mutually agreeable. When type i acquires good i, he immediately consumes it, produces a new unit of good i+1, and stores it until the next date, when he meets a new trading partner. Hence, in equilibrium type i always enters a trading period with either good i+1 or good i+2, and never good i. This means that $p(t) = [p_1(t),p_2(t),p_3(t)]$, where $p_i(t)$ is the proportion of type i agents holding their production good i+1 at date t, completely describes the distribution of inventories at a point in time. A steady-state inventory distribution satisfies $p(t) = p$ for all t.

Agents choose strategies for deciding when to trade, given the strategies of others and p. In this section, as in Kiyotaki and Wright (1989), we consider only time-invariant pure-strategies, which we define by functions τ_i, where $\tau_i(j,k) = 1$ if type i wants to trade good j for good k, and $\tau_i(j,k) = 0$ otherwise. Also as in Kiyotaki and Wright (1989), we assume $\tau_i(j,k) = 0$ if and only if $\tau_i(k,j) = 1$ (type i either prefers good k to good j or vice-versa). Under mild conditions to guarantee that agents do not want to drop out of the economy, we can also show that $\tau_i(j,i) = 1$ for all j (type i always trades for his own consumption good). Therefore, each type i's strategy is completely specified once we decide whether $\sigma_i = \tau_i(i+1,i+2)$ is 1 or 0. If $\sigma_i = 1$, then i is willing to trade his production good i+1 for the intermediate good i+2, which he later uses to buy his consumption good; if $\sigma_i = 0$, then i keeps his production good until he can trade for his consumption good directly. Choosing σ_i therefore amounts to choosing whether or not type i is willing to use good i+2 as a medium of exchange.

A symmetric, steady-state, pure-strategy equilibrium is defined to be a vector of inventories $p = (p_1,p_2,p_3)$ and a vector of strategies $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ such that: (1) when agents use strategies σ, the resulting steady-state inventory distribution is p, and (2) for all i, σ_i maximizes expected discounted utility from consumption net of storage costs, given σ and p. The following result from Kiyotaki and Wright (1989) describes the set of such equilibria for Model A. In order to reduce notation in our statement of this result, we normalize utility, with no loss in generality, so that $\beta u/3 = 1$. Given this, it turns out that everything depends on the single parameter $\delta_1 = c_{13}-c_{12}$.

Proposition 1. In Model A, if $\delta_1 \geq 1/2$, then $\sigma = (0,1,0)$ is the unique symmetric, steady-state, pure-strategy equilibrium; if $\delta_1 \leq \sqrt{2} - 1$, then $\sigma = (1,1,0)$ is the unique such equilibrium; if $\sqrt{2} - 1 < \delta_1 < 1/2$, then there exists no such equilibrium.

The intuition behind Proposition 1 is simple. Assume that $\sigma_2 = 1$ and $\sigma_3 = 0$, and consider the best response problem of a type 1 agent. (It is easy to show that $\sigma_2 = 1$ and $\sigma_3 = 0$ are best responses for type 2 and type 3, for all parameter values, when either $\sigma_1 = 0$ or 1.) The instantaneous cost to type 1 of trading good 2 for good 3 is δ_1, the increase in one-period storage disutility. The instantaneous benefit is the increase in the probability of meeting someone with good 1 next period who is willing to trade, $[p_3 - (1-p_2)]/3$, times the discounted utility of consumption, βu. Now $\sigma_1 = 1$ if and only if the cost is less than the benefit – that is, if and only if $\delta_1 \leq p_3 - (1-p_2)$. But p_2 and p_3 depend on strategies. Simple calculations show that $\sigma = (0,1,0)$ implies $p_3 - (1-p_2) = 1/2$, and so $\sigma_1 = 0$ is the best response as long as $\delta_1 \geq 1/2$ (if good 3 is much more costly to store than good 2, type 1 opts for direct barter rather than using a medium of exchange). Also, $\sigma = (1,1,0)$ implies $p_3 - (1-p_2) = \sqrt{2} - 1$, and so $\sigma_1 = 1$ is the best response as long as $\delta_1 \leq \sqrt{2} - 1$ (if good 3 is not too much more costly to store, type 1 opts to use it as a medium of exchange).

If $\sqrt{2} - 1 < \delta_1 < 1/2$, then no symmetric, steady-state, pure-strategy equilibrium exists. Thus, when all type 1 agents refuse to accept good 3, type 2 agents end up holding more of good 3 and less of good 1, which means type 1 agents ought to accept good 3 to facilitate trade with type 3. On the other hand, when all type 1 agents accept good 3, type 2 agents end up holding less of good 3 and more of good 1, which means type 1 agents do not need to trade with type 3 and ought to refuse to accept good 3. Clearly, what is needed is to have some, but not all, type 1 agents accept good 3, or equivalently, to have type 1 agents accept good 3 with a probability strictly between 0 and 1. We analyze this possibility in Section 4. Another alternative would be to have type 1 agents accept good 1 at some dates and not others, a possibility we consider in Section 7.

3. The Generalized Model

Let $s_i(t)$ be the probability that type i plays strategy $\sigma_i = 1$ at date t, and let $s(t) = [s_1(t),s_2(t),s_3(t)]$. If the probability of agent i trading good i+1 for good i+2 at date t is $s_i(t)$, then the probability of him trading good i+2 for good i+1 is $1-s_i(t)$. This implies that whether an agent prefers good i+1 or good i+2 at date t does not depend on the good with which he enters the period. Given s(t), the "trading matrices" in Figure 1 depict the probability of exchange in any particular meeting, excluding cases where two agents of the same type meet, because we assume with no loss in generality that i never trades with his own type.

Given any path for the strategy vector s(t), the transition equation for the inventory distribution is given by $p(t+1) = \gamma[s(t),p(t)]$, where $\gamma:\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$. We can use the trading matrices to compute the explicit functional form of γ, which we write as

$$(3.1) \qquad \gamma(s,p) = p - \tfrac{1}{3}G(s,p),$$

where $G(s,p) = [G_1(s,p),G_2(s,p),G_3(s,p)]$, and

$$(3.2) \qquad G_i(s,p) = p_i p_{i+1} s_i - (1-p_i)[(1-p_{i+2})(1-s_i)+p_{i+2}+(1-p_{i+1})(1-s_{i+1})].$$

Type 1 \ Type 2	3	1
2	s_1	1
3	0	$1\text{-}s_2$

Type 1 meets type 2

Type 2 \ Type 3	1	2
3	s_2	1
1	0	$1\text{-}s_3$

Type 2 meets type 3

Type 3 \ Type 1	2	3
1	s_3	1
2	0	$1\text{-}s_1$

Type 3 meets type 1

Figure 1: Trading matrices

The first term on the right-hand side of (3.2) represents (three times) the measure of type i agents who switch from good i+1 to good i+2 through exchange, while the second term represents (three times) the measure of type i agents who switch from good i+2 to good i+1 through exchange, consumption and production.

We now describe the individual decision problem. Let $V_{ij}(t)$ be the expected discounted utility at the end of period t for type i given an inventory of good j (the payoff, or value, function). If we define $\Delta_i(t) = V_{i,i+1}(t) - V_{i,i+2}(t)$, the maximizing choice of $s_i(t)$, which is generally a correspondence, satisfies

(3.3) $$s_i(t) \in \begin{cases} \{0\} & \text{if } \Delta_i(t) > 0 \\ [0,1] & \text{if } \Delta_i(t) = 0. \\ \{1\} & \text{if } \Delta_i(t) < 0 \end{cases}$$

For example, for type 1, if $\Delta_1(t) = V_{12}(t) - V_{13}(t) > 0$ then he should set $s_1(t) = 0$ and not trade good 2 for good 3; if $\Delta_1(t) < 0$ then he should set $s_1(t) = 1$ and trade good 2 for good 3 whenever he can; and if $\Delta_1(t) = 0$ he is indifferent and may choose the probability $s_1(t)$ to be anything in [0,1].

To illustrate the technique, we explicitly derive $\Delta_1(t)$. Consider a type 1 agent with good 2 at the end of period t. He first pays his storage cost c_{12} and, next period, he meets an agent of type 1, 2, or 3, each with probability 1/3. If he meets another type 1, he does not trade and leaves with $V_{12}(t+1)$. Now suppose he meets a type 2, who will always want to trade given our type 1 agent holds good 2. With probability $p_2(t+1)$, the type 2 agent has good 3 and our agent chooses the probability of trade $s_1(t+1)$, while with probability $1-p_2(t+1)$ the type 2 agent has good 1, there is a double coincidence of wants, and our agent definitely trades, consumes and produces a new unit of good 2. Now suppose he meets a type 3. With probability $1-p_3(t+1)$ both agents have good 2 and they cannot trade, while with probability $p_3(t+1)$ the type 3 agent has good 1 and our agent wants to trade, so type 3 chooses the probability $s_3(t+1)$.

Writing this explicitly, the payoff for type 1 with good 2 at t is

$$\begin{aligned}V_{12}(t) = -c_{12} + \frac{\beta}{3}\Big[&V_{12}(t+1) + p_2(t+1)\{s_1(t+1)V_{13}(t+1)+[1-s_1(t+1)]V_{12}(t+1)\}\\ &+ [1-p_2(t+1)][u+V_{12}(t+1)] + [1-p_3(t+1)]V_{12}(t+1)\\ &+ p_3(t+1)\{s_3(t+1)[u+V_{12}(t+1)]+[1-s_3(t+1)]V_{12}(t+1)\}\Big].\end{aligned}$$

Minor simplification yields

$$\begin{aligned}V_{12}(t) = -c_{12} + 1-p_2(t+1) + p_3(t+1)s_3(t+1) + \beta V_{12}(t+1)\\ - \frac{\beta}{3}p_2(t+1)s_1(t+1)\Delta_1(t+1),\end{aligned}$$

using our normalization $\beta u/3 = 1$. A similar analysis for the case where type 1 has good 3 at t yields

$$\begin{aligned}V_{13}(t) = -c_{13} + [1-p_2(t+1)][1-s_2(t+1)] + p_3(t+1) + \beta V_{13}(t+1)\\ + \frac{\beta}{3}\{[1-p_2(t+1)][1-s_2(t+1)]+1-[1-p_3(t+1)]s_1(t+1)\}\Delta_1(t+1).\end{aligned}$$

Subtracting these two equations, we arrive at

$$\begin{aligned}\Delta_1(t) = \delta_1 + [1-p_2(t+1)]s_2(t+1) - p_3(t+1)[1-s_3(t+1)]\\ + \beta\Big[1 - \frac{1}{3}\{p_2(t+1)s_1(t+1)+[1-p_2(t+1)][1-s_2(t+1)]\\ + p_3(t+1) + [1-p_3(t+1)][1-s_1(t+1)]\}\Big]\Delta_1(t+1).\end{aligned}$$

By symmetry, for any type i, we can write

$$(3.4)\qquad \Delta_i(t) = F_i[s(t+1),p(t+1),\delta] + \beta Q_i[s(t+1),p(t+1)]\Delta_i(t+1),$$

where $\delta_i = c_{i,i+2} - c_{i,i+1}$, $\delta = (\delta_1,\delta_2,\delta_3)$, and we define

$$(3.5)\qquad F_i(s,p,\delta) = \delta_i + (1-p_{i+1})s_{i+1} - p_{i+2}(1-s_{i+2})$$

$$(3.6)\qquad Q_i(s,p) = 1 - \frac{1}{3}[p_{i+1}s_i+(1-p_{i+1})(1-s_{i+1})+p_{i+2}+(1-p_{i+2})(1-s_i)].$$

It will be convenient below to write $\Delta(t) = [\Delta_1(t),\Delta_2(t),\Delta_3(t)]$ and $F(s,p) = [F_1(s,p),F_2(s,p),F_3(s,p)]$. Given any path for $[s(t),p(t)]$, a path for $\Delta_i(t)$ satisfying (3.4) implies the maximizing choice of $s_i(t)$ at every date via the best response condition (3.3). Notice that $[s(t),p(t)]$ does not pin down the sequence $\Delta_i(t)$, however, without some condition on $\Delta_i(0)$, and without $\Delta_i(0)$ there is nothing to pin down the initial choice of $s_i(0)$ in the model; this will be important in Section 6.

In any case, we now have the following definition. A (symmetric) equilibrium, given an initial distribution of inventories $p(0)$, is a path $[s(t),p(t),\Delta(t)]$ such that: (1) given strategies $s(t)$, $p(t)$ satisfies the transition equation (3.1) for all t; (2) given $[s(t),p(t)]$, $\Delta_i(t)$ and $s_i(t)$ satisfy the best response conditions (3.3) and (3.4) for all t. A steady-state equilibrium can be defined by $[s(t),p(t),\Delta(t)] = (s,p,\Delta)$ for all t, such that: (1) given s, p is a fixed point of the transition equation; (2) given (s,p), Δ_i and s_i satisfy the best response conditions. Notice the definition of a steady-state equilibrium does not make reference to a given initial condition $p(0)$. Also notice that, when $\Delta(t) = \Delta$ for all t, (3.4) implies

$$[1-\beta Q_i(s,p)]\Delta_i = F_i(s,p,\delta). \tag{3.7}$$

Since $\beta Q_i(s,p) < 1$, all that matters for the best response condition in steady-state is the sign of $F_i(s,p,\delta)$.

To close this section, we point out that no mention has been made so far of whether we are considering what was called Model A or Model B in Kiyotaki and Wright (1989). The relevant difference between the two models is simply that Model A assumes that $\delta_i > 0$ for two of the three agent types, while Model B assumes that $\delta_i > 0$ for only one of the three agent types (the reordering of production-consumption specializations used in the original paper is merely a relabeling, given the δ_i parameters are reordered appropriately). Hence, the above specification incorporates both Model A and Model B, depending on the parameter vector δ.

4. Steady-State Equilibria

In this section and the next, we restrict attention to steady-state equilibria. Our immediate goal is to fill in the gap in Proposition 1 by constructing a symmetric, steady-state, mixed-strategy equilibrium in the region of parameter space for which we know that no symmetric equilibria in pure-strategies exist for Model A.

We will construct a steady-state equilibrium in which $s_2 = 1$, $s_3 = 0$, and $s_1 \in [0,1]$ will be determined as a function of the parameters. Because Model A specifies that $\delta_1 > 0$, $\delta_2 < 0$, and $\delta_3 > 0$, this implies that type 2 always trade their production good 3 for the lower storage cost good 1, type 3 never trade their production good 1 for the higher storage cost good, and type 1 may or may not trade their production good 2 for the higher storage cost good 3. That this may be an equilibrium is suggested by the observations at the end of Section 2; it is also suggested by Theorem 7 in Aiyagari and Wallace (1989) (see also Theorem 7 in Gintis 1989), which says that there always exists a mixed-strategy, steady-state equilibrium in fairly general versions of this model in which every agent always accepts the lowest storage cost good (good 1 in this case).

If we substitute $s_2 = 1$ and $s_3 = 0$ into (3.5), then the functions F_i reduce to

$$F_1(s,p,\delta) = \delta_1 + (1-p_2) - p_3$$

$$F_2(s,p,\delta) = \delta_2 - p_1(1-s_1)$$

$$F_3(s,p,\delta) = \delta_3 + (1-p_1)s_1.$$

For any $\delta_2 < 0$ we have $F_2 < 0$, which implies $s_2 = 1$ is a best response for type 2; similarly, for any $\delta_3 > 0$ we have $F_3 > 0$, which implies $s_3 = 0$ is a best response for type 3. For type 1, the sign of F_1 depends on p. Solving for the steady-state inventory distribution $p = \gamma(s,p)$ as a function of s_1, we find

$$p = \left[\sqrt{1+s_1}/(1+s_1),\ (\sqrt{1+s_1}-1)/s_1,\ 1\right]$$

(this holds for $s_1 > 0$; for $s_1 = 0$, take the limit). Hence,

$$F_1 = \delta_1 - (\sqrt{1+s_1}-1)/s_1.$$

A strategy s_1 is a best response as long as it satisfies condition (3.3). Combinations of δ_1 and s_1 consistent with (3.3) are easily computed to be

$$\delta_1 \le \sqrt{2}-1 \text{ and } s_1 = 1$$

$$\sqrt{2}-1 \le \delta_1 \le 1/2 \text{ and } s_1 = (1-2\delta_1)/\delta_1^2$$

$$1/2 \le \delta_1 \text{ and } s_1 = 0,$$

as shown in Figure 2. Hence, for all $\delta_1 > 0$ an equilibrium exists, and this fills in the gap in Proposition 1. Notice that the two pure-strategy equilibria in Proposition 1 reappear for appropriate values of δ_1, and are connected by mixed-strategy equilibria. We also point out that it is equivalent here to reinterpret our symmetric mixed-strategy equilibria as nonsymmetric pure-strategy equilibria, where the fraction s_1 of type 1 agents play strategy $\sigma_1 = 1$ with probability 1 while the fraction $1-s_1$ play $\sigma_1 = 0$ with probability 1.

We now have existence of a steady-state equilibrium for all parameter values in Model A. Introducing mixed-strategies leads to other new possibilities as well. For example, suppose $s_1 = 1$, $s_2 = 0$, and $s_3 \in (0,1)$. It is not difficult to check $F_1 < 0$, $F_2 > 0$, and $F_3 = 0$, so that these are indeed best responses, if and only if the following conditions are satisfied:

$$\sqrt{2}-1 < \delta_3 < 1/2 \text{ and } s_3 = (1-2\delta_3)/\delta_3^2$$

$$\delta_1 \le (\delta_3^2+2\delta_3-1)/(\delta_3-\delta_3^2)$$

$$\delta_2 \ge -(1-2\delta_3)^2/(\delta_3^2-\delta_3^3).$$

Hence, in a certain region of parameter space, this is an equilibrium for Model A. But, from Proposition 1, there already exists an equilibrium with

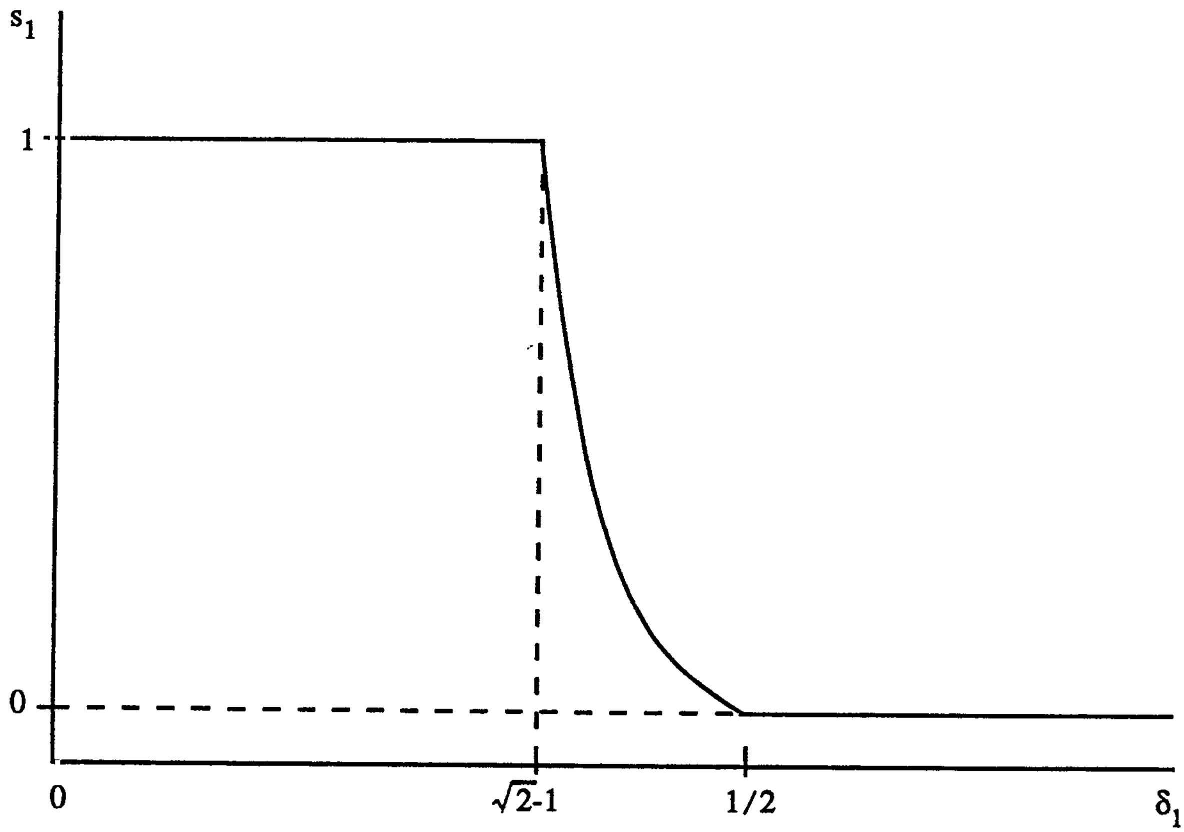

Figure 2: Equilibrium value of s_1 as a function of δ_1

$s_1 = 1$, $s_2 = 1$, and $s_3 = 0$, as long as $\delta_1 \leq \sqrt{2} - 1$. For a nonempty open set of parameter values, these equilibria exist simultaneously.

Interestingly, the two equilibria have rather different properties. In the equilibrium described in Proposition 1, type 1 agents use good 3 as a medium of exchange and type 2 agents use good 1 as a medium of exchange, while type 3 agents hold onto their production good until they can trade directly for their consumption good. In the new equilibrium, type 1 agents still use good 3 as a medium of exchange, but now type 2 agents hold onto their production good until they can trade directly for their consumption good, and type 3 agents respond by sometimes accepting good 2 to facilitate trade with type 2. In this new equilibrium, agents never trade a high storage cost good for a lower storage cost good (unless the latter is their consumption good, of course). This is another example of how extrinsic beliefs can have a large impact in determining which objects serve as media of exchange, even when the objects have different intrinsic properties. In the next section, however, we show that the number of steady-state equilibria is finite. Hence, it is not the case that anything is possible, and so extrinsic beliefs concerning which objects will play the role of money are not overwhelming.

5. Generic Finiteness of Steady-State Equilibria

To prove generic finiteness of the number of steady-state equilibria, we utilize the transversality theorem of differential topology (see Guillemin and Pollack 1974, pp. 68-69, or Hirsch 1976, pp. 74-77). Similar results have been obtained for finite n-person normal form games by Harsanyi (1973) and van Damme (1983). Because of the interaction of p and s in our payoff functions, however, their results do not apply directly to our model.

We use the following notation: If $F(x,\alpha)$ is a function of a vector of variables x and a vector of parameters α, then we write $f_\alpha(x) = F(x,\alpha)$ for fixed α.

Transversality Theorem. Let $F: X\times A \to Y$, where $X\subset\mathbb{R}^\ell$ is contained in the closure of an open set and $A\subset\mathbb{R}^m$ and $Y\subset\mathbb{R}^n$ are open sets. Suppose that F is continuously differentiable of order r, where $r > \max(\ell-n,0)$. Suppose too that, if $(x,\alpha)\in X\times A$ satisfies $F(x,\alpha) = 0$, then $DF(x,\alpha)$ has rank n. Then $f_\alpha(x) = 0$ implies $Df_\alpha(x)$ has rank n for all α in a subset of A of full Lebesgue measure.

Notice that, if $\ell < n$, then the $n\times\ell$ matrix $Df_\alpha(x)$ cannot possibly have rank n. The conclusion of the theorem, in this case, is that for all α in a subset of A of full Lebesgue measure, there is no $x\in X$ such that $f_\alpha(x) = 0$.

The intuition behind this theorem is one of counting equations and unknowns. If $\ell < n$, then there are more equations than unknowns; we therefore would not expect there to be any solutions if we have sufficient freedom to perturb the equations. If $\ell = n$, then there are the same numbers of equations and unknowns; we therefore would expect any solutions to the equations to be locally unique if we have sufficient freedom to perturb the equations. Indeed, if the $n\times n$ matrix $Df_\alpha(\bar{x})$ has rank n, then the inverse function theorem tells us that the solution $\bar{x}$ to the equation $f_\alpha(\bar{x}) = 0$ is locally unique. If $\ell > n$, then there are more unknowns than equations; we would therefore expect there to be an infinite number of solutions. Although we could use this theorem and the implicit function theorem to count degrees of freedom and parameterize the set of solutions, we are concerned here only with situations where $\ell = n$. In this case, the formal criterion for sufficient freedom to perturb the equations is that the $n\times(\ell+m)$ matrix $DF(x,\alpha) = [D_1F(x,\alpha), D_2F(x,\alpha)]$ has rank n whenever $F(x,\alpha) = 0$.

Applying this theorem to our commodity money economy, we are forced to consider different cases: $0 < s_i < 1$ for all i; $s_i = 0$ for some i; and $s_i = 1$ for some i. In each case, equilibria are solutions to a different set of equations. In the case where $0 < s_i < 1$ for all i, equilibria are solutions to $E(s,p,\delta) = 0$ where $E:\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3$ is given by the rule $E(s,p,\delta) = [F(s,p,\delta),G(s,p)]$ and F and G are given as in Section 3. In cases where $s_i = 0$ or $s_i = 1$ for some i, we replace equation i of $F(s,p,\delta) = 0$ with $s_i = 0$ or $s_i-1 = 0$. Therefore, here we solve $E(s,p,\delta) = 0$, where $E(s,p,\delta) = [\bar{F}(s,p,\delta),G(s,p)]$ and $\bar{F}$ is formed appropriately. In all of these cases, $(s,p)\in\mathbb{R}^3\times\mathbb{R}^3$ is the vector of endogenous variables and $\delta\in\mathbb{R}^3$ is the vector of parameters. Letting

$$D = \{\delta\in\mathbb{R}^3 | \delta_1 > 0,\ \delta_2 < 0\}$$

be the set of parameters, we allow for the specification of either Model A or Model B (after a relabeling) simply by allowing δ_3 to be either positive or negative.

Proposition 2. For all δ in a set of full Lebesgue measure in D, there is a finite number of steady-state equilibria.

Proof. The strategy of proof is simple: We first apply the transversality theorem to each of the cases discussed above, where (s,p) is allowed to range over a set contained in the closure of an open set. We then use the inverse function theorem to show that solutions to each system of equations are locally unique for all δ in a set of full Lebesgue measure in D. Finally, we restrict (s,p) to a compact set and argue that the number of equilibria is, in fact, finite.

All equilibria lie in the set

$$S = \{(s,p)\in\mathbb{R}^3\times\mathbb{R}^3 | 0 \le s_i \le 1,\ 0 \le p_i \le 1\},$$

which is obviously compact. Observe that there can be no equilibrium in which $p_i = 0$ for some i, since the steady-state condition $p = \gamma(s,p)$ implies that in this case $p_1 = p_2 = p_3 = 0$ and $s_1 = s_2 = s_3 = 1$, which cannot be best responses for any $\delta\in D$. Indeed, no matter what p is, $s_1 = s_2 = s_3 = 1$ cannot be best responses. In applying the transversality theorem, we therefore restrict our attention to the set

$$S' = \{(s,p)\in S | 0 < p_i \text{ for all } i,\ s_i < 1 \text{ for some } i\},$$

which is contained in the closure of the open set defined by letting all the inequalities be strict.

First, consider the case where $E(s,p,\delta) = [F(s,p,\delta),G(s,p)]$, which corresponds to equilibria where $0 < s_i < 1$ for all i. In this case,

$$DE(s,p,\delta) = \begin{bmatrix} D_1F(s,p,\delta) & D_2F(s,p,\delta) & D_3F(s,p,\delta) \\ D_1G(s,p) & D_2G(s,p) & 0 \end{bmatrix}.$$

Notice $D_3F(s,p,\delta)$ is the 3×3 identity matrix. If we could show that the 3×3 matrix $D_1G(s,p)$ has rank three, we would then know that $DE(s,p,\delta)$ has rank six. The transversality theorem would imply that, for all δ in a set of full Lebesgue measure, the 6×6 matrix

$$De_\delta(s,p) = \begin{bmatrix} D_1f_\delta(s,p) & D_2f_\delta(s,p) \\ D_1G(s,p) & D_2G(s,p) \end{bmatrix}$$

has rank 6 whenever $e_\delta(s,p) = 0$. The inverse function theorem would then imply that any such solution has an open neighborhood in $\mathbb{R}^3\times\mathbb{R}^3$ in which it is the only solution. To see that $D_1G(s,p)$ has rank three, we merely compute it:

$$D_1G(s,p) = \begin{bmatrix} p_1p_2+(1-p_1)(1-p_3) & (1-p_1)(1-p_2) & 0 \\ 0 & p_2p_3+(1-p_2)(1-p_1) & (1-p_2)(1-p_3) \\ (1-p_3)(1-p_1) & 0 & p_3p_1+(1-p_3)(1-p_2) \end{bmatrix}.$$

Since this matrix has sign pattern

$$\begin{bmatrix} + & + \text{ or } 0 & 0 \\ 0 & + & + \text{ or } 0 \\ + \text{ or } 0 & 0 & + \end{bmatrix},$$

it has rank three, as required.

The cases where we replace $F_i(s,p,\delta) = 0$ with $s_i = 0$ or $s_i-1 = 0$ are more complicated. We can still argue that $D\overline{F}(s,p,\delta)$ has rank three because it contains three linearly independent columns, those columns in $D_1\overline{F}$ for which $\overline{F}_i(s,p,\delta)$ equals s_i or s_i-1 and those columns in $D_3\overline{F}$ for which $\overline{F}_i(s,p,\delta)$ equals $F_i(s,p,\delta)$. We now need to find three linearly independent columns in $[D_1G(s,p),D_2G(s,p)]$ that do not include the columns in $D_1G(s,p)$ for which $\overline{F}_i(s,p,\delta)$ equals s_i or s_i-1.

The case where $s_i = 0$ for some i is the easiest. Here, we can argue that the matrix

$$D_2G(s,p) =$$

$$\begin{bmatrix} \begin{matrix} p_2s_1+(1-p_2)(1-s_2) \\ +p_3+(1-p_3)(1-s_1) \end{matrix} & p_1s_1+(1-p_1)(1-s_2) & -(1-p_1)s_1 \\ -(1-p_2)s_2 & \begin{matrix} p_3s_2+(1-p_3)(1-s_3) \\ +p_1+(1-p_1)(1-s_2) \end{matrix} & p_2s_2+(1-p_2)(1-s_3) \\ p_3s_3+(1-p_3)(1-s_1) & -(1-p_3)s_3 & \begin{matrix} p_1s_3+(1-p_1)(1-s_1) \\ +p_2+(1-p_2)(1-s_3) \end{matrix} \end{bmatrix}$$

has rank three because it has sign pattern

$$\begin{bmatrix} + & + \text{ or } 0 & - \text{ or } 0 \\ - \text{ or } 0 & + & + \text{ or } 0 \\ + \text{ or } 0 & - \text{ or } 0 & + \end{bmatrix},$$

and at least one of the potentially negative elements is zero.

In cases where $s_i-1 = 0$ for some i, we need to consider combinations of columns from D_1G and D_2G. When $s_i-1 = 0$, for example, we choose the second and third column from D_1G and the first column from D_2G. These columns form a 3x3 matrix with sign pattern

$$\begin{bmatrix} + \text{ or } 0 & 0 & + \\ 0 & + \text{ or } 0 & - \text{ or } 0 \\ 0 & + & + \text{ or } 0 \end{bmatrix},$$

which has rank three. The cases where $s_2-1 = 0$ or $s_3-1 = 0$ are similar. In the cases where $s_i-1 = 0$ for two i, we combine one column form D_1G with two from D_2G. We have already ruled out the case where $s_i-1 = 0$ for all i.

Now consider the set of pairs $(s,p)\in S'$ that satisfy any of the various combinations of equations $E(s,p,\delta) = 0$. This set includes the set of equilibria, but may be larger since there is no guarantee that the appropriate inequality in the best response condition (3.3) is satisfied if $s_i = 0$ or $s_i-1 = 0$. The set of δ such that all of the solutions to these equations are locally unique has full Lebesgue measure since it is the intersection of a finite number (the number of possible cases) of sets of full Lebesgue measure. Consequently, for almost all δ the set of equilibria consists of locally unique points.

Suppose now that we allow (s,p) to range over all of S. Could there be an infinite number of equilibria? If there were, then there would be a convergent subsequence of equilibria since S is compact. There would then be two possibilities: this convergent subsequence could converge to $(s,p)\in S'$ or it could converge to (s,p) for which $s_1 = s_2 = s_3 = 1$. If $(s,p)\in S'$, then it too is an equilibrium, but it would not be locally unique, which is a contradiction. If $s_1 = s_2 = s_3 = 1$, any sequence converging to (s,p) would eventually violate the best response conditions. Consequently, for all δ in a subset of D with full Lebesgue measure, there is a finite number of steady-state equilibria. □

Several extensions of this result are possible. First, since S is compact, we could argue that the set of δ for which the number of equilibria is finite is also open. Second, it is easy to see that for almost all δ, the inequalities in the best response condition (3.3) must be strict if $s_i = 0$ or $s_i = 1$; otherwise, the equilibrium would be a solution to a system with more equations than unknowns. Third, using an index theorem, we could argue that for almost all δ the number of equilibria is odd (see, for example, Mas-Colell 1985).

6. Dynamic Equilibria

We now turn our attention to equilibria in which the strategies s(t) and inventories p(t) vary over time. In this section, we show that there may be a robust continuum of dynamic equilibria, in contrast to the case of steady-states. Although many different types of dynamic equilibria are possible, for simplicity, we will restrict attention for now to the case where $0 < s_i(t) < 1$ for all i and t, and look for dynamic equilibria that converge to a steady-state (the construction was suggested by an example in Aiyagari and Wallace 1990, in a two-agent model with fiat currency).

The fact that $0 < s_i(t) < 1$ for all t requires $\Delta_i(t) = 0$ for all t, or $F_i[s(t),p(t),\delta] = 0$ for all t, by equation (3.4). Notice that the condition $F[s(t),p(t),\delta] = 0$ is actually linear in s(t), and can be written

$$\begin{bmatrix} 0 & 1-p_2(t) & p_3(t) \\ p_1(t) & 0 & 1-p_3(t) \\ 1-p_1(t) & p_2(t) & 0 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} p_3(t)-\delta_1 \\ p_1(t)-\delta_2 \\ p_2(t)-\delta_3 \end{bmatrix}.$$

As argued in the previous section, $p_i > 0$ for all i in any steady state equilibrium. Thus, if we start with $p_i(0) > 0$ sufficiently close to a stable steady state, then $p_i(t) > 0$ for all t and the above equation can be solved to yield $s(t) = \varphi[p(t)]$ for all t (given a fixed δ). In particular, suppose that $\Delta(t+1) = 0$ and every agent i is indifferent between goods i+1 and i+2 at date t+1. Then we can choose s(t+1) arbitrarily, and if we choose $s(t+1) = \varphi[p(t+1)]$, subject to the condition $s_i(t+1) \in (0,1)$ for all i, this guarantees $\Delta(t) = 0$. In other words, if agents are willing to randomize at t+1, then as long as we choose $s(t+1)$ appropriately, they will also be willing to randomize at t.

We can use this logic to construct a continuum of dynamic equilibria, given the initial inventory distribution $p(0)$ (which is fixed by nature). First, choose $s(0)$ so that $s_i(0) \in (0,1)$ for all i. Given $[s(0),p(0)]$, the transition equation implies $p(1) = \gamma[s(0),p(0)]$. Now set $s(1) = \varphi[p(1)]$, so that $F[s(1),p(1),\delta] = 0$, which means that $\Delta(0) = 0$ as long as $\Delta(1) = 0$, and our original choices of $s_i(0)$ are indeed best responses as long as $\Delta(1) = 0$. Continuing in this manner, $p(2) = \gamma[s(1),p(1)]$, and we can set $s(2) = \varphi[p(2)]$ to guarantee that $\Delta(1) = 0$ as long as $\Delta(2) = 0$. This implies the transition dynamics for $p(t)$,

$$p(t+1) = \gamma[\varphi(p(t)),p(t)] = T[p(t)].$$

Notice that $s(0)$ is not pinned down in any way here. Consequently, since $p(1) = \gamma[s(0),p(0)]$, even given $p(0)$, $p(1)$ is not pinned down in this system.

Any path satisfying $p(t+1) = T[p(t)]$ and $s(t) = \varphi[p(t)]$ for all t is an equilibrium with all agents mixing, as long as $0 < s_i(t) < 1$ for all t. A steady-state $(\bar{s},\bar{p})$ solves $\bar{p} = T(\bar{p})$ and $\bar{s} = \varphi(\bar{p})$. The linearization of $T(\cdot)$ around a steady-state $\bar{p}$ is

$$[p(t+1)-\bar{p}] = DT(\bar{p})[p(t)-\bar{p}],$$

where $DT = D_1\gamma D\varphi + D_2\gamma$. We are interested in constructing an example where all of the eigenvalues of the 3x3 matrix $DT(\bar{p})$ are less than one in modulus. Given such an example, we can use the implicit function theorem and the local stable manifold theorem (see Irwin 1980) to argue that, for all $[s(0),p(0)]$ in an open neighborhood of $(\bar{s},\bar{p})$, there exists an equilibrium path $[s(t),p(t)]$ satisfying $p(t+1) = T[p(t)]$ and $s(t) = \varphi[p(t)]$.

For one such example, consider the economy where $\delta = (0.05,-0.05,0.05)$. It has a steady-state $(\bar{s},\bar{p})$ with

$$\bar{s} = (0.7270,0.5538,0.6850),\ \bar{p} = (0.6349,0.7070,0.6740).$$

At this steady-state, the eigenvalues of $DT(\bar{p})$ are

$$\lambda = 0.2151,\ 0.5021+0.0850i,\ 0.5021-0.0850i,$$

each of which is less than one in modulus. Hence, the local stable manifold theorem implies that, for all $p(1)$ in some open set containing $\bar{p}$, $p(t+1) = T[p(t)]$ converges to $\bar{p}$. It is easy to verify that the conditions of the implicit function theorem are satisfied. Thus, for all $[s(0),p(0),p(1)]$ in some open set containing $(\bar{s},\bar{p},\bar{p})$, the vector $s(0)$ satisfying $p(1) = \gamma[s(0),p(0)]$ varies continuously with $[p(0),p(1)]$. Furthermore, for fixed $p(0)$, the implicit function $s(0) = \psi[p(0),p(1)]$ is an invertible function of $p(1)$. Since $p(1)$ can vary over an open set and still produce a path that converges to $\bar{p}$, $s(0)$ can also vary over an open set and, together with a fixed $p(0)$, produce a dynamic equilibrium that converges to $(\bar{s},\bar{p})$.

Figure 3 shows the dynamic path of the above example beginning from $p(0) = (0.2,0.9,0.8)$ and $s(0) = (0.5,0.5,0.5)$. Given $p(0)$, the system converges for a fairly wide range of $s(0)$, although for other values of $s(0)$ it does not, and eventually some or all $s_i(t)$ leave $[0,1]$. This is true for a fairly wide range of $p(0)$. Finally, this example of a continuum of equilibria is robust. It is easy to verify that the 6x6 matrix $De_\delta(s,p)$ has

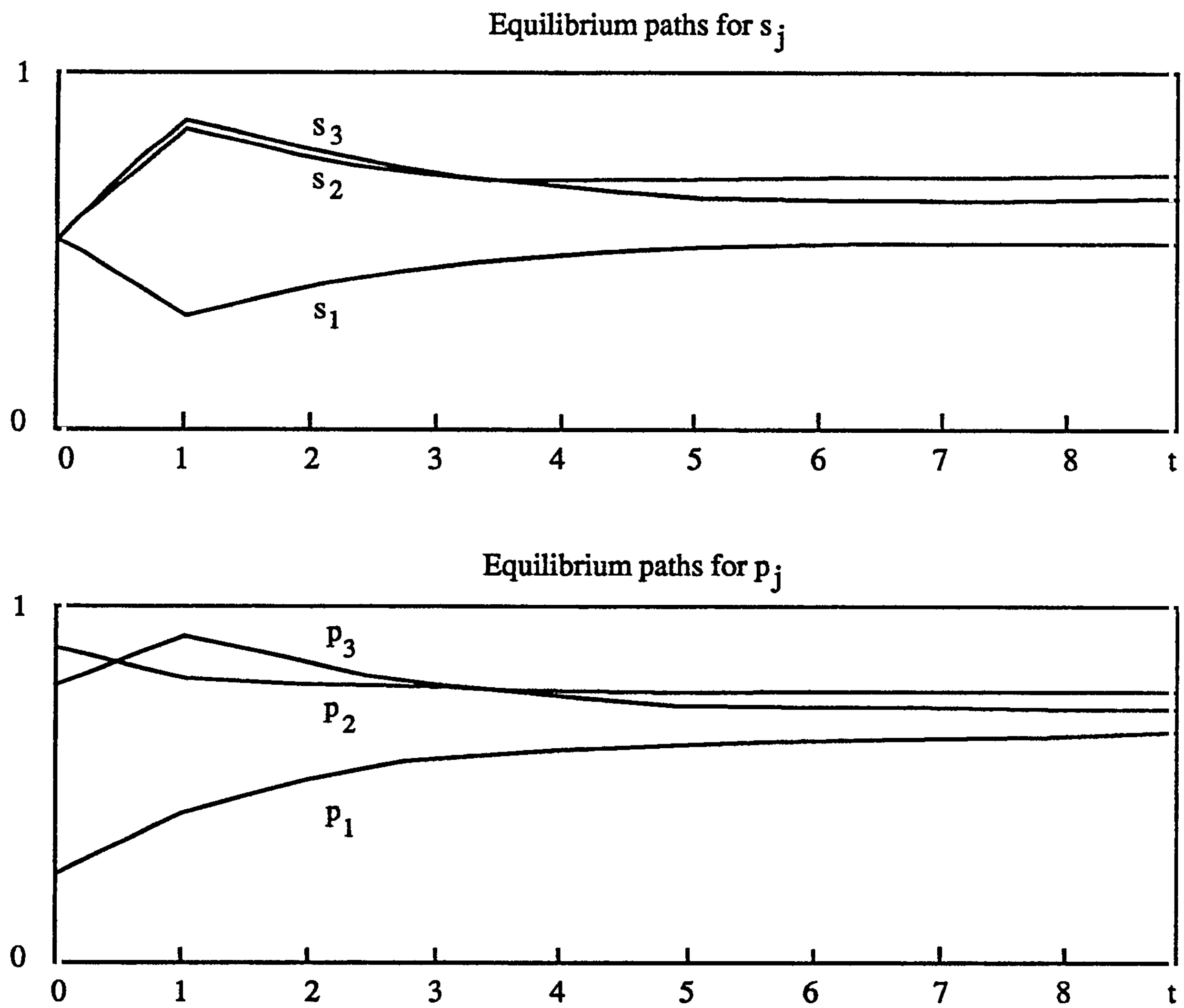

Figure 3: Equilibrium paths for strategies and inventories

full rank. Consequently, the parameters $\delta = (0.05,-0.05,0.05)$ constitute a regular economy, and the implicit function theorem implies that the steady-state equilibrium $(\overline{s},\overline{p})$ varies continuously with δ. Small perturbations in δ produce small perturbations in the matrix $DT(\overline{p})$, and the continuity of the eigenvalues in the elements of this matrix therefore implies that small perturbations in δ still yield economies in which all three eigenvalues are less than one in modulus. Hence, all economies with δ close to $(0.05,-0.05,0.05)$ will display qualitatively the same three dimensional indeterminacy.

Although we do not present the details here, it is easy to produce examples with a lower dimension of indeterminacy. Suppose, for example, DT has two eigenvalues less than 1 and the third greater than 1 in modulus. Then the local stable manifold theorem says that there is a two dimensional manifold of inventories $p(1)$ near $\overline{p}$ that lead to convergence to $\overline{p}$. The implicit function theorem implies that, for fixed $p(0)$, this corresponds to a two dimensional manifold of initial strategies $s(0)$. Similarly, it may be possible to produce examples with no indeterminacy or with one dimension of indeterminacy; everything depends on the number of stable eigenvalues of DT. Furthermore, in this section we have only considered dynamic equilibria where $s_i(t)\in(0,1)$ for all i and for all t. One could also consider dynamic equilibria where some types use pure-strategies while others use mixed-strategies, or where a given type fluctuates between strategies. We take a step in this direction in the next section.

7. Cyclic Equilibria

Here we construct a dynamic equilibrium where $s_2(t) = 1$ for all t, $s_3(t) = 0$ for all t, and $s_1(t) = 1$ if t is odd, $s_1(t) = 0$ if t is even, which is a two-period cycle. In this equilibrium, $\Delta_1(t)$ will fluctuate between positive and negative, and so type 1 agents will be willing to trade good 2 for good 3 in one period but not the next. (Note that, when $\Delta_1 > 0$, they are not willing to dispose of good 3 and produce a new unit of good 2, as long as we assume that there is a cost to doing so that is prohibitively high.) Given these strategies, it is easy to confirm that the dynamical system $p(t+1) = \gamma[s(t),p(t)]$ converges to a two-cycle: $p(t) = p^e$ if t is even, $p(t) = p^o$ if t is odd, where

$$p^e = (0.7741, 0.4531, 1.0),\ p^o = (0.8494, 0.4432, 1.0).$$

We now verify that, for certain parameter values δ and inventories $p(t)$, these strategies are best responses. It is easy to check this for types 2 and 3 by showing that $F_2[s(t),p(t),\delta] < 0$ and $F_3[s(t),p(t),\delta] > 0$ in both even and odd periods, as long as $\delta_2 < 0$ and $\delta_3 > 0$. For type 1,

$$\Delta_1(t) = \delta_1 - p_2(t+1) + \frac{\beta}{3}[2-p_2(t+1)s_1(t+1)]\Delta_1(t+1).$$

In a two-period cycle, $\Delta_1(t) = \Delta^e$ if t is even and $\Delta_1(t) = \Delta^o$ if t is odd; that is,

$$\Delta^e = \delta_1 - p_2^o + \frac{\beta}{3}(2-p_2^o s_1^o)\Delta^o,$$

$$\Delta^o = \delta_1 - p_2^e + \frac{\beta}{3}(2-p_2^e s_1^e)\Delta^e.$$

This can be solved to yield

$$\Phi\Delta^e = \delta_1 - p_2^o + \frac{\beta}{3}(2-p_2^o s_1^o)(\delta_1-p_2^e),$$

$$\Phi\Delta^o = \delta_1 - p_2^e + \frac{\beta}{3}(2-p_2^e s_1^e)(\delta_1-p_2^o),$$

where $\Phi > 0$. The cyclic strategy $s_1(t) = 0$ if t is even and $s_1(t) = 1$ if t is odd is a best response for type 1 as long as $\Delta^e \geq 0 \geq \Delta^o$. Since s_1^e, s_1^o, p_2^e and p_2^o are known, these inequalities depend only on δ_1 and β, and Figure 4 shows the region of (β,δ_1) space in which $\Delta^e \geq 0$ and $\Delta^o \leq 0$ both hold. In this region, all of the equilibrium conditions are satisfied.

The same procedure can be used to construct equilibrium cycles of other periodicities, and Figure 4 also shows the region of (β,δ_1) space in which there exists a three-cycle equilibrium, with $s_2(t) = 1$ for all t, $s_3(t) = 0$ for all t, and $s_1(t) = (0,0,1,0,0,1,\ldots)$. Notice both of these cyclical equilibria exist only for values of δ_1 that do not allow a steady-state equilibrium in pure-strategies – that is, only for $\sqrt{2} - 1 < \delta_1 < 1/2$. Finally, we point out that these cycles are stable. Given cyclic strategies, p(t) locally converges to a cyclic distribution, and as long as $\Delta_1(t)$ alternates in sign in the right way in the limit, it will alternate in sign close to the limit. Thus, the cyclic strategies will also be best responses in the neighborhood of the limit cycle. Given p(0) (determined by nature), there will be cyclic strategies that imply that p(t) converges to a limit cycle, and in certain regions of parameter space these strategies are best responses along the entire path.

8. Concluding Remarks

In this paper, we have generalized the simple commodity money model in Kiyotaki and Wright (1989) by introducing mixed-strategies and dynamic equilibria. We have constructed some mixed-strategy, steady-state equilibria, including equilibria for regions of parameter space for which no pure-strategy, steady-state equilibria exist, and have proved the generic finiteness of the set of steady-state equilibria. We have also described

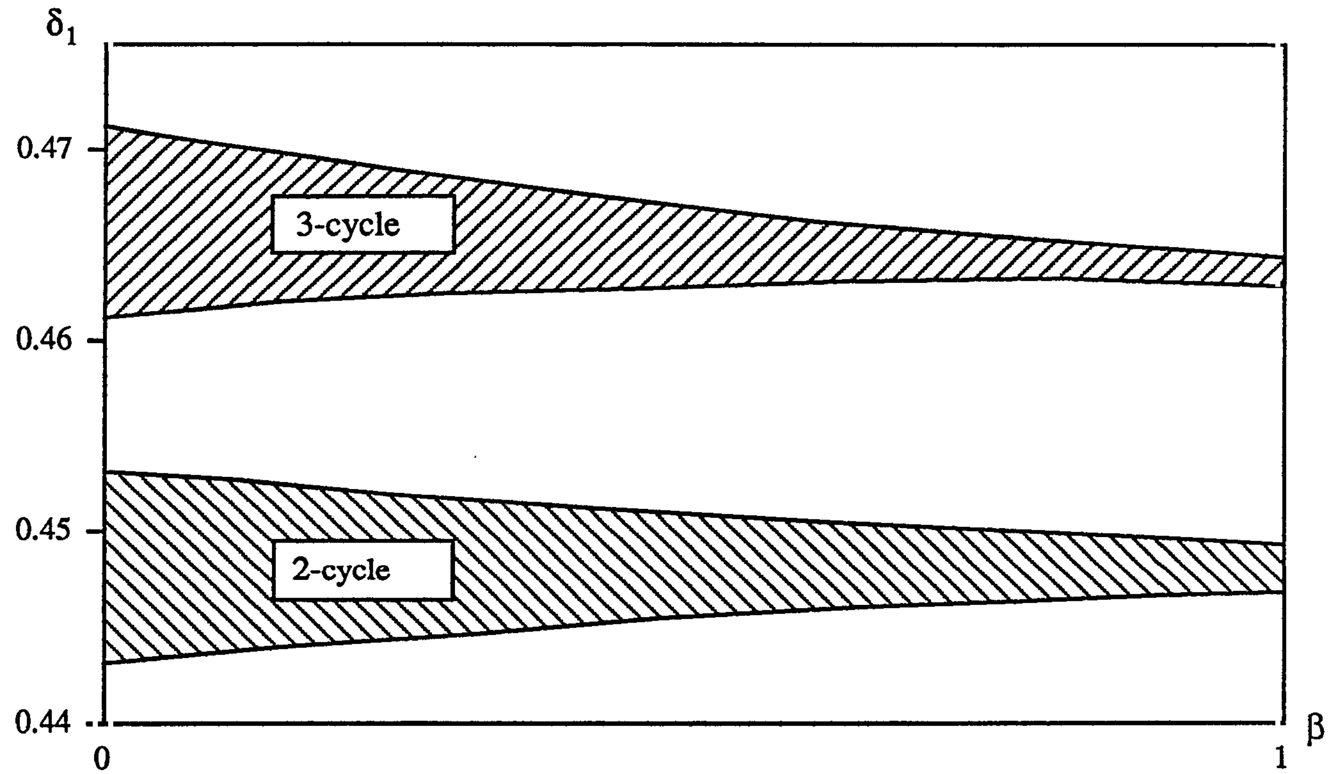

Figure 4: Regions where cycles exist

some interesting dynamic equilibria. As pointed out in the introduction, this model shares several properties that have been established for overlapping generations models, and it can be analyzed using very similar techniques.

One issue not addressed here is the existence of an equilibrium for arbitrary initial inventory distributions. It is straightforward to prove the existence of such an equilibrium by adapting the approach used for overlapping generations models (see, for example, Balasko, Cass and Shell 1980): truncate the economy at some finite date $\bar{t}$ by arbitrarily choosing the value functions $V_{ij}(\bar{t})$, prove existence for the finite economy using a standard fixed point argument, let $\bar{t}$ tend to infinity, and take the appropriate limits. However, as pointed out in Aiyagari and Wallace (1989), the hard part in this model is establishing that the equilibrium discovered in this way is not a degenerate equilibrium. An extension of the technique they use for steady state equilibria would have to be used to show that dynamic equilibria are nondegenerate at least for certain parameter values.

Another issue not addressed here is the existence of sunspot equilibria, where the equilibrium strategies and hence commodity monies fluctuate randomly over time, even though the fundamentals of the model are deterministic and time invariant. Such equilibria come up often in overlapping generations models, of course; see Azariadis and Geusnerie (1986), for example. Sunspot equilibria have been constructed for models that are simpler, but similar to the one used here in the sense that agents search for partners and trade requires a double coincidence of wants, in Kiyotaki and Wright (1990) and in Burdett and Wright (1991). Exploring the possible relevance of sunspots in this environment is left to future research.

References

Aiyagari, S.R., and N. Wallace (1989) "Existence of steady states with positive consumption in the Kiyotaki - Wright model," *Review of Economic Studies*, forthcoming.

Aiyagari, S.R., and N. Wallace (1990) "Fiat money in the Kiyotaki - Wright model," Federal Reserve Bank of Minneapolis Working Paper 453.

Azariadis, C. and R. Geusnerie (1986) "Sunspots and cycles," *Review of Economic Studies* 53, 725-738.

Balasko, Y., D. Cass and K. Shell (1980) "Existence of competitive equilibrium in a general overlapping generations model," Journal *of Economic Theory* 23, 307-22.

Benhabib, J., and R.H. Day (1982) "A characterization of erratic dynamics in the overlapping generations model," *Journal of Economic Dynamics and Control* 4, 37-55.

Burdett, K. and R. Wright (1991) "Two-sided search," unpublished manuscript, Federal Reserve Bank of Minneapolis.

van Damme, E. (1983) *Refinements of the Nash Equilibrium Concept*. Lecture Notes in Economics and Mathematical Systems, No. 219. Berlin: Springer-Verlag.

Gintis, H. (1989) "Money in general equilibrium," unpublished manuscript, University of Massachusetts - Amherst.

Grandmont, J.-M. (1985) "On endogenous competitive business cycles," *Econometrica* 53, 995-1043.

Guillemin, V., and A. Pollack (1974) *Differential Topology*. Englewood Cliffs, N.J.: Prentice-Hall.

Harsanyi, J.C. (1973) "Oddness of the number of equilibrium points: A new proof," *International Journal of Game Theory* 2, 235-50.

Hirsch, M.W. (1976) *Differential Topology*. New York: Springer-Verlag.

Irwin, M.C. (1980) *Smooth Dynamical Systems*. New York: Academic Press.

Jovanovic, B. and R.W. Rosenthal (1988) "Anonymous Sequential Games," *Journal of Matematical Economics* 17, 77-87.

Kehoe, T.J. (1985) "Multiplicity of equilibria and comparative statics," *Quarterly Journal of Economics* 100, 119-47.

Kehoe, T.J., and D.K. Levine (1984) "Regularity in overlapping generations exchange economies," *Journal of Mathematical Economics* 13, 69-93.

Kehoe, T.J., and D.K. Levine (1985). "Comparative statics and perfect foresight in infinite horizon models," *Econometrica* 53, 433-53.

Kiyotaki, N., and R. Wright (1989) "On money as a medium of exchange," *Journal of Political Economy* 97, 927-54.

Kiyotaki, N., and R. Wright (1990) "Search for a theory of money," NBER Working Paper No. 3482.

Mas-Colell, A. (1985) *The Theory of General Economic Equilibrium: A Differentiable Approach*. Cambridge: Cambridge University Press.