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UNIQUENESS OF EQUILIBRIUM IN LARGE-SCALE NUMERICAL GENERAL EQUILIBRIUM MODELS

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1. Introduction

In a topological space of general equilibrium models there is a non-empty open set of economies with unique equilibria and another non-empty open set of economies with multiple equilibria; neither circumstance is degenerate. Recent work on uniqueness of equilibrium in economies with production and with taxes has provided index theorems along with explicit formulae for calculation of the index of an equilibrium [Kehoe (1980, 1985a)]. Since the index of every equilibrium is positive if and only if the equilibrium is unique, these formulae provide necessary and sufficient mathematical conditions for uniqueness. Unfortunately, these conditions usually lose their sufficiency when we attempt to translate them into economically interpretable restrictions on the parameters of a model. It remains an open problem whether or not non-uniqueness is largely a theoretical curiosum, similar in practical significance to the possibility of the simplex algorithm requiring an exponential number of steps to solve a linear programming problem, or whether it is a prevalent feature even in empirically based general equilibrium models. The difficulty is that it is next to impossible to establish uniqueness of equilibrium for large dimensional models since no algorithm is known that finds all of the equilibria of a model.

In this note we report on some calculations made for the Fullerton, King, Shoven and Whalley (1981) general equilibrium model of the U.S. economy and tax system that clearly demonstrate that the equilibrium is unique. The method used to establish uniqueness is to take the mapping of the unit simplex from three to two dimensions through numerical elimination while

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guaranteeing the eliminated excess demand has a unique solution. In this way the system of excess demand functions that define the model can be numerically evaluated for the unit interval and can be plotted. The plots clearly reveal uniqueness since only one intersection with the zero axis occurs. We can also report on similar calculations for a model of the Mexican economy developed by Serra-Puche (1983) and Kehoe and Serra-Puche (1983). This model involves one more relative price than that of Fullerton et al. and so requires a two-dimensional grid search. We discuss the possibilities for extending the methodology to even higher dimensional problems and the difficulties that would arise.

Non-uniqueness has been a lurking issue ever since work on applied general equilibrium models began in the early 1970s. If more than one equilibrium solution exists, the modeller can have little confidence in any policy appraisal from his analysis since there is no obvious way to decide which of the equilibria the economy would go to. Earlier work has used ad hoc procedures to try to find other equilibria (such as approaching an equilibrium from different directions and/or at different speeds, or displacing the algorithm from the solution to see if it returns) and not produced any examples of non-uniqueness. Kehoe (1985b), however, has produced a numerical example with seemingly innocuous parameter specifications for Cobb–Douglas preferences and linear activities with widely separated multiple equilibria. The calculations presented here provide some comfort for modellers since in at least two cases it seems clearly established that uniqueness occurs. In neither case is any general condition known to be sufficient for uniqueness satisfied.

2. The Fullerton–King–Shoven–Whalley tax model

The Fullerton–King–Shoven–Whalley tax model is a dynamic sequenced general equilibrium model of the U.S. economy and tax system that incorporates the major tax distortions of goods and factor markets in the U.S. Single period equilibria are sequenced through capital stock accumulation reflecting price endogenous savings decisions in each period. In any single period, there are 19 producer good industries that use capital and labor in constant elasticity of substitution (CES) value-added production functions. They also use the outputs of other industries through an input–output matrix with fixed coefficients. Tax rates on labor for each industry are derived by taking payroll taxes as a proportion of labor income, while tax rates on capital for each industry are derived by taking corporate income and property taxes as a proportion of capital income. Each of these 19 producer goods are used directly by government, for export, and for investment goods, but indirectly for consumption through a fixed-coefficient correspondence matrix linking 15 consumer goods with the 19 producer goods. This procedure is necessary because the goods classification of consumer expenditure data is different from the output of the 19 production sectors.

There are 12 consumer groups, differentiated by income class, each with an initial endowment of capital and labor. Consumers maximize utility defined over 15 current consumer goods, current leisure, and anticipated future consumption. The price of future consumption depends inversely on the expected rate of return, which equals the current rate of return. The resulting demand for future consumption is translated into current desired savings. The latter, in turn, is translated into a vector of investment demands for the 19 industry outputs. Government collects taxes from both the production and demand sides of the economy and uses the revenue to purchase producer goods via Cobb–Douglas demand functions, to make direct transfer payments to consumers and to subsidize government enterprises. A simple trade sector closes the model.

The equilibrium conditions of the model are then used to determine behavioral equation parameters consistent with a 1973 benchmark data set. This procedure calibrates the model in the sense that benchmark data can be reproduced as an equilibrium solution to the model before any policy changes are considered.

To implement this procedure exogenous elasticities of substitution between capital and labor are specified in each industry, along with labor supply and savings elasticities to which substitution elasticities in preferences are calibrated. Factor employments by industry are used to derive production function weights, and expenditure data are used to derive utility function weights. This ensures that the behavior of agents is mutually consistent in benchmark data before policy changes are considered. Through their interaction, utility-maximizing consumers and profit-maximizing producers reach a single period competitive equilibrium where all profits are zero and supply equals demand for each good and factor. Single period equilibria are sequenced through endogenous savings decisions that augment the capital stock of the economy. An exogenous labor force growth rate is assumed.

A balanced growth path is calculated that begins with the replicated data, has constant prices, and implies quantities that all grow at the labor force growth rate. Tax parameters are then altered and a revised sequence of equilibria calculated. A higher net return to capital, for instance, implies more savings, capital growth, and a transition with an asymptotic approach to a new balanced growth path. The model assumes no involuntary unemployment of factors. Markets are perfectly competitive with no externalities, quantity constraints, or barriers to factor mobility. Since a complete set of prices and quantities are computed under alternative tax policies, the change in national income, utility or income changes for each group, and all new factor allocations among industries can be used in policy appraisal.

3. Establishing uniqueness of equilibrium

The essential structure of the Fullerton–King–Shoven–Whalley model involves a three-dimensional mapping of the unit simplex into itself for each single period equilibrium. The three dimensions involve the excess demands for capital and labor and the government budget imbalance. We write this system as

$$\begin{aligned} &\xi_L(w,r,R)-\bar{L}=0,\\ &\xi_K(w,r,R)-\bar{K}=0, \qquad (1)\\ &R-T(w,r,R)=0, \end{aligned}$$

where ξ_L and ξ_K are the derived demand functions for capital and labor corresponding to the factor prices w and r and the level of revenue disbursements by government R. T defines government tax revenues at any (w, r, R).

This three-dimensional system defines the model: Knowing the prices of factors, we can use cost functions and competitive zero-profit conditions to compute the prices of commodities. Knowing the prices of commodities and revenues dispersed, we can determine household incomes and commodity demands. Knowing commodity demands, we can use the requirement that supply equals demand for commodities to derive demands for the factors. Derived factor demands yield the reduced form above along with a government budget balance equation. None of the excess demands in (1) exist in analytical form but can be numerically evaluated.

We can eliminate R from the system (1) by calculating the zero of the government budget imbalance correspondence for each (w,r). Since no marginal tax rate in the model is greater than one, it follows that $R-T(w,r,R)$ is strictly increasing in R. Consequently, for any given (w,r), we can solve for the value of R, $R(w,r)$, that gives government budget balance and substitute into the factor demand functions in the knowledge that the associated value of R is unique. This reduces (1) to a system involving two excess demand functions:

$$\begin{aligned} &\xi_L(w,r,R(w,r))-\bar{L}=0,\\ &\xi_K(w,r,R(w,r))-\bar{K}=0, \end{aligned} \qquad (2)$$

which can then be evaluated over the unit interval where $0 \leqq r \leqq 1$ and $w=1-r$.

In fig. 1 we plot the values of these functions for the central case

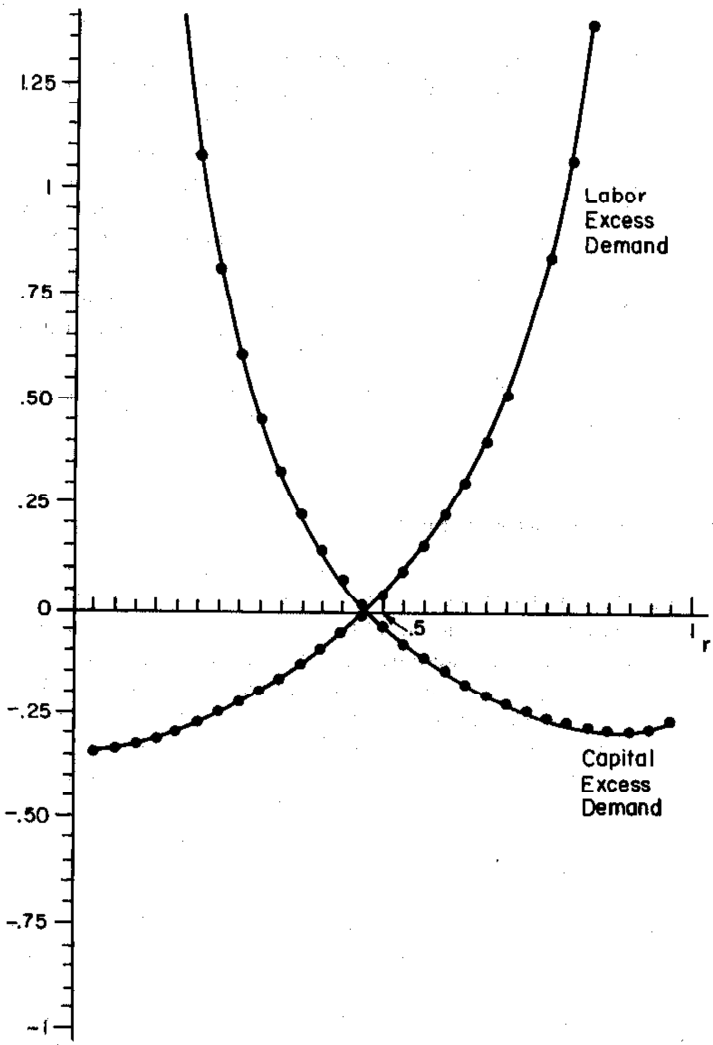

Fig. 1

specification of the model used by Fullerton et al. for one of the policy changes they consider. The calculations reported here are for only one equilibrium in the sequence. We normalize the excess demands so that the largest excess demand is 10.0 at the tail of the unit interval to make the plot manageable. We consider 29 function evaluations corresponding to a subdivision of the unit interval in which r increases by increments of 1/30. The choice of a grid with a denominator of 30, although somewhat arbitrary, reflects a trade-off between a desire for accuracy and computational costs.

Fig. 1 indicates uniqueness of equilibria through the single intersection of the excess demand functions with the zero axis. We have also confirmed uniqueness in other specifications of this model where production function elasticities are changed in all industries. While it is undoubtedly the case that it is possible to select parameters for this model so as to engineer a case of non-uniqueness, there seems to be substantial comfort for the modellers in these results.

It is important to know not only that the equilibrium is unique in the base case but that it is unique after any parameter change in a comparative statics exercise. Since changes in the parameters of the model perturb the functions in (2), and therefore the graph in fig. 1, continuously, it is reasonable to believe that only a fairly drastic parameter change would result in non-uniqueness of equilibrium.

4. Applications in higher dimensions

Using the methodology of the previous section, we are also able to establish that the model of the Mexican economy of Kehoe and Serra-Puche has a unique equilibrium. The structure of this model is similar to that of Fullerton et al. except that there are three factors of production: rural labor, urban labor, and capital. In addition to the relative prices of these three goods, a level of government tax revenues must be determined by the equilibrium conditions. As in the previous section, this variable can be eliminated numerically. This still leaves a simplex of three factor prices, a two-dimensional set, to search over for equilibria.

The government in this model is allowed to run a deficit by selling bonds, which the consumers regard as perfect substitutes for physical capital as savings instruments. There is, however, still a requirement that government receipts, including those from bond sales, equals government expenditure. This requirement results in equilibrium conditions similar to that in the model of Fullerton et al.

One problem with an exhaustive search in more than one dimension is that it is not easy to identify the zeros of the excess demand functions. In one dimension, if a function changes sign over a small interval, continuity implies that it must be zero somewhere on the interval. In higher dimensions there is no analogous result. Accordingly, a finer grid search is required to allow a reasonable approximation to an equilibrium. Unfortunately, for a fixed grid size, the number of points goes up exponentially with the dimension of the problem. Changing prices by increments of 1/30, we have to search over a grid with 406 points, rather than the 29 points required for the corresponding one-dimensional search. While a search over a simplex involving four prices would be costly, but computationally feasible, a search over a simplex involving five prices would be prohibitively time consuming.

The restriction of this technique therefore makes it inappropriate for models with many factors of production, those with sector specific capital, for example. The technique can, however, be easily applied to models that allow substitution in intermediate production since it is still the case that commodity prices can be calculated once factor prices are specified.

The search methodology employed in this paper may suggest directions for future research into equilibria of higher dimensional models with the kind of decomposability properties described by Mansur and Whalley (1982). Suppose the equilibrium conditions for a model can be written:

$$f(p, q^1, \ldots, q^m) = 0,$$
$$g^i(p, q^i) = 0, \quad i = 1, \ldots, m. \tag{3}$$

Here f and p are vectors of the same dimension and g^i and q^i are vectors of the same dimension. The f refers to excess demands for common goods, and the g^i, $i = 1, \ldots, m$, refer to excess demand for non-common goods, arranged in m blocks. If we can show that, for any vector p, the vector q^i that satisfies $g^i(p, q^i) = 0$ is unique, then we can eliminate each of the variables $q^1, \ldots, q^m$ numerically. To establish uniqueness, we then only need to find all the solutions to the system:

$$f(p, q^1(p), \ldots, q^m(p)) = 0,$$

which may be of low enough dimension to be computationally feasible. International trade models with traded and non-traded goods are one class of models to which this methodology can be applied. Intertemporal models with durable and non-durable goods and some kind of intertemporal separability of preferences may be another.

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